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Supersymmetric approach for generating quasi-exactly solvable potentials with arbitrary two known eigenstates

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Abstract
Using supersymmetric quantum mechanics we construct the quasi-exactly solvable (QES) potentials with arbitrary two known eigenstates. The QES potential and the wavefunctions of the two energy levels are expressed by some generating function the properties of which determine the state numbers of these levels. Choosing different generating functions we present a few explicit examples of the QES potentials.

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1. Introduction

From the early days of quantum mechanics there has been continual interest in the models for which the corresponding Schrödinger equation can be solved exactly. The number of totally exactly solvable potentials is rather limited. Therefore, recently much attention has been given to the quasi-exactly solvable (QES) potentials for which a finite number of energy levels and corresponding wavefunctions are known in explicit form.

The first examples of QES potentials were given in [1–4]. Subsequently several methods were worked out for generating QES potentials and as a result many QES potentials were established [5–13]. One of the methods is the generation of new QES potentials using supersymmetric (SUSY) quantum mechanics [13–15] (for a review of SUSY quantum mechanics see [16]). The idea of the SUSY method for constructing QES potentials is the following. Starting from some initial QES potential with $n + 1$ known eigenstates and using the properties of the unbroken SUSY one obtains the SUSY partner potential which is a new QES potential with the n known eigenstates.

In our recent paper [17] we proposed a new SUSY method for generating QES potentials with two known eigenstates. This method, in contrast to the one in [13–15], does not require knowledge of the initial QES potential in order to generate a new QES one. General expressions for the superpotential, the potential energy and two wavefunctions which correspond to two energy levels were obtained. Within the frame of this method we have obtained QES potentials for which we have found in the explicit form the energy levels and wavefunctions of the ground and first excited states. One should mention here also [18] in which the general expression for the QES potentials with two known eigenstates was obtained without resorting to SUSY quantum mechanics (see also a very recent paper by Dolya and Zaslavskii [19]). Although this method is direct and simpler than the SUSY approach the latter still has some advantages: namely, the SUSY method developed in [17] can be extended for the generation of QES potentials with three known eigenstates [20] and conditionally exactly solvable (CES) potentials [21]. The CES potentials are those for which the eigenvalue problem for the corresponding Hamiltonian is exactly solvable only when the potential parameters obey certain conditions [22]. About using SUSY quantum mechanics for the construction of the CES potentials see [23–25]. It is also worth mentioning the very recent [26] in which the authors established the connection between the SUSY approach for constructing QES potentials with two known eigenstates [17, 21] and the Turbiner approach [5].

Note that the general expressions for QES potential and corresponding wavefunctions derived by Dolya and Zaslavskii in [19] without resorting to SUSY quantum mechanics are the same as those obtained in [17, 21] using the SUSY method. In [17, 21] we have used these general expressions for constructing QES potentials with the ground and first excited states. A new interesting result obtained by Dolya and Zaslavskii is that they have shown that it is possible to obtain not only the ground and first excited states but any pair of the energy levels and the corresponding wavefunctions.

The aim of this paper is to extend the SUSY method proposed in our papers [17, 21] for constructing QES potentials with arbitrary two known eigenstates. In [17, 21] we used nonsingular superpotentials and obtained QES potentials with explicitly known ground and first excited states. In this paper using singular superpotentials we obtain nonsingular QES potentials for which we know in explicit form any pair of the energy levels and the corresponding wavefunctions. Of course, the idea of using singular superpotential in SUSY quantum mechanics is well known (see [16] for a review). Nevertheless, new in this paper is that in the frame of SUSY quantum mechanics with singular superpotentials we derive nonsingular QES potentials with arbitrary two known eigenstates.

2. SUSY quantum mechanics and QES problems

In the Witten model of SUSY quantum mechanics the SUSY partner Hamiltonians $H_\pm$ read

$$H_\pm = B^\mp B^\pm = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x) \tag{1}$$

where

$$B^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right) \tag{2}$$

$$V_\pm(x) = \tfrac{1}{2}(W^2(x) \pm W'(x)) \qquad W'(x) = \frac{\mathrm{d}W(x)}{\mathrm{d}x} \tag{3}$$

and $W(x)$ is referred to as a superpotential. In this paper we consider the systems on the full real line $-\infty < x < \infty$.

We study the eigenvalue problem for the Hamiltonian H_-

$$B^+B^-\psi_E^-(x) = E\psi_E^-(x). \tag{4}$$

Applying to the left- and right-hand sides of this equation the operator B^- we obtain the equation for the eigenvalue problem of the Hamiltonian H_+

$$B^- B^+(B^- \psi_E^-(x)) = E(B^- \psi_E^-(x)) \tag{5}$$

from which it follows that

$$\psi_E^+(x) = C B^- \psi_E^-(x) \tag{6}$$

is the solution of the eigenvalue problem for the Hamiltonian H_+ with energy E, and $C = 1/\sqrt{E}$ is the normalization constant if the wavefunction $\psi^+(x)$ is square integrable. It is also possible that $\psi^+(x)$ does not satisfy the necessary conditions and does not belong to the eigenfunctions of the Hamiltonian H_+. Such a situation occurs for singular superpotentials (see review [16]). Nevertheless, the function $\psi^+(x)$ is the solution of (5). Applying to (6) the operator B^+ we obtain

$$\psi_E^-(x) = C B^+ \psi_E^+(x). \tag{7}$$

Now let us analyse the eigenvalue problem for the Hamiltonian H_-. Due to the factorization of the Hamiltonian the wavefunction of the zero-energy state satisfies $B^- \psi_0^- = 0$ and reads

$$\psi_0^-(x) = C_0^- \exp\left(- \int W(x)\, \mathrm{d}x \right) \tag{8}$$

where C_0^- is the normalization constant. In order to satisfy the condition of the square integrability of the wavefunction (8) we put

$$\mathrm{sign}(W(\pm\infty)) = \pm 1. \tag{9}$$

We are interested in the potential energy $V_-(x)$ free of singularities. The simplest way to satisfy this condition is to consider a superpotential $W(x)$ which is free of singularities. Then $\psi_0^-(x)$ corresponds to the zero-energy ground state of the Hamiltonian H_-. Just this case was considered in our paper [17]. But it is also possible to get a nonsingular potential energy $V_-(x)$ using a singular superpotential. Let us assume that $W(x)$ has the simple poles at the points x_k with the following behaviour in the vicinity of x_k:

$$W(x) = \frac{A_{-1}}{x - x_k} + A_0 + A_1(x - x_k) + \mathrm{O}((x - x_k)^2). \tag{10}$$

Then $V_-(x)$ in the vicinity of x_k reads

$$2V_-(x) = \frac{A_{-1}(A_{-1}+1)}{(x - x_k)^2} + 2\frac{A_{-1}A_0}{x - x_k} + 2A_{-1}A_1 + A_0^2 - A_1 + \mathrm{O}(x - x_k). \tag{11}$$

The first case $A_{-1} = 0$ leads to the nonsingular $W(x)$ and $V_-(x)$. The second case $A_{-1} = -1$ and $A_0 = 0$ gives the nonsingular potential energy $V_-(x)$ with a singular superpotential. Here it is worth stressing that the SUSY partner potential energy in this case is singular with the following behaviour in the vicinity of x_k: $2V_+(x) = 2/(x - x_k)^2 - A_1 + \mathrm{O}(x - x_k)$.

Let us analyse the second case. Using equation (8) we obtain the behaviour of the wavefunction in the vicinity of the point x_k as follows: $\psi_0^-(x) \sim |x - x_k|(1 - A_1(x - x_k)^2/2)$. As we see, $\mathrm{d}\psi_0^-(x)/\mathrm{d}x$ is a discontinuous function at the points x_k. In order to obtain the wavefunction with continuous derivative we use the simple fact that if $\psi_0^-(x)$ in the domain $x_k < x < x_{k+1}$ satisfies the Schrödinger equation then the function with the opposite sign $-\psi_0^-(x)$ satisfies the same equation too. Thus, we can change the sign of wavefunction in the domains (x_k, x_{k+1}) in such a way that $\psi_0^-(x)$ and its derivative $\mathrm{d}\psi_0^-(x)/\mathrm{d}x$ will be continuous functions. In fact this means that it is necessary to make the substitution $|f| \to f$. Then the behaviour of the wavefunction in the vicinity of x_k is

$$\psi_0^-(x) \sim (x - x_k)(1 - A_1(x - x_k)^2/2) \tag{12}$$

and the wavefunction has zeros at the points x_k. Thus, the zero-energy wavefunction has n nodes (n is the number of poles of the superpotential) and corresponds to the nth excited state. Note that in this case the ground state energy is less than zero. Choosing different superpotentials $W(x)$ we can easily construct different QES potentials with one known eigenstate.

In contrast to this the construction of the QES potentials with two known eigenstates is not a trivial problem. One state of the Hamiltonian H_- with zero energy is known and is given by (8). In order to obtain one more state of H_- we use the following well known procedure exploited in SUSY quantum mechanics. Let us consider the SUSY partner of H_-, i.e. the Hamiltonian H_+. If we calculate some state of H_+ we immediately find a new excited state of H_- using transformation (7). In order to calculate some state of H_+ let us rewrite it in the following form:

$$H_+ = H_-^{(1)} + \epsilon = B_1^+ B_1^- + \epsilon \qquad \epsilon > 0 \tag{13}$$

which leads to the relation between the potential energies

$$V_+(x) = V_-^{(1)}(x) + \epsilon \tag{14}$$

and superpotentials

$$W^2(x) + W'(x) = W_1^2(x) - W_1'(x) + 2\epsilon \tag{15}$$

where ϵ is the energy of the state of H_+ since we supposed that $H_-^{(1)}$ similarly to H_- has the zero-energy state, and $B_1^\pm$ and $V_-^{(1)}(x)$ are given by (2) and (3) with the new superpotential $W_1(x)$.

As we see from (13) the wavefunction of H_+ with energy $E = \epsilon$ is also the zero-energy wavefunction of $H_-^{(1)}$ and satisfies $B_1^- \psi_\epsilon^+(x) = 0$. The solution of this equation is

$$\psi_\epsilon^+(x) = C^+ \exp\left(- \int W_1(x)\,\mathrm{d}x \right). \tag{16}$$

Using (7) we obtain the wavefunction of the excited state with energy level $E = \epsilon$ for the Hamiltonian H_-

$$\psi_\epsilon^-(x) = C^-\ W_+(x) \exp\left(- \int W_1(x)\,\mathrm{d}x \right) \tag{17}$$

where we have introduced the notation $W_+(x) = W_1(x) + W(x)$. In order to satisfy the square integrability of this function at infinity we impose the superpotential $W_1(x)$ with the same condition as $W(x)$ (9). Then $W_+(x)$ satisfies the same condition (9) too.

In order to obtain the explicit expression for the wavefunction $\psi_0^-(x)$ with zero energy and the wavefunction $\psi_\epsilon^-(x)$ with energy ϵ given by (8) and (17) it is necessary to obtain the explicit expression for superpotentials $W(x)$ and $W_1(x)$. This is the subject of the next section.

3. Solutions for superpotentials and construction of nonsingular QES potentials

The superpotentials $W(x)$ and $W_1(x)$ satisfy equation (15). Note that (15) is the Riccati equation which cannot be solved exactly with respect to $W_1(x)$ for a given $W(x)$ and vice versa. But we can find such a pair of $W(x)$ and $W_1(x)$ that satisfies equation (15). For this purpose let us rewrite equation (15) in the following form:

$$W_+'(x) = W_-(x) W_+(x) + 2\epsilon \tag{18}$$

where

$$W_+(x) = W_1(x) + W(x) \tag{19}$$

$$W_-(x) = W_1(x) - W(x). \tag{20}$$

This new equation can be easily solved with respect to $W_-(x)$ for a given $W_+(x)$ and vice versa. In this paper we use the solution of equation (18) with respect to $W_-(x)$

$$W_-(x) = (W_+'(x) - 2\epsilon)/W_+(x). \tag{21}$$

Then from (19) to (21) we obtain the pair of $W(x)$, $W_1(x)$ that satisfies equation (15):

$$W(x) = \tfrac{1}{2}(W_+(x) - (W_+'(x) - 2\epsilon)/W_+(x)) \tag{22}$$

$$W_1(x) = \tfrac{1}{2}(W_+(x) + (W_+'(x) - 2\epsilon)/W_+(x)) \tag{23}$$

where $W_+(x)$ is some function of x which generates the superpotentials $W(x)$ and $W_1(x)$. Let us stress that $W(x)$, $W_1(x)$ and $W_+(x)$ must satisfy condition (9). It is necessary to note that the general solutions (22), (23) of equation (15) was obtained earlier in [27] in the context of paraSUSY quantum mechanics.

In our earlier paper [17] we considered only the superpotentials free of singularities $W(x)$ and $W_1(x)$. To satisfy a nonsingularity of the superpotentials we considered a continuous function $W_+(x)$ that has only one simple zero. Because $W_+(x)$ was considered as a continuous function which satisfies condition (9) the function $W_+(x)$ must have at least one zero. Then, as we see from (21) to (23), $W_-(x)$, $W(x)$ and $W_1(x)$ have poles. In order to construct the superpotentials free of singularities we supposed that $W_+(x)$ has only one simple zero at $x = x_0$. In this case the pole of $W_-(x)$ and $W(x)$, $W_1(x)$ at $x = x_0$ can be cancelled by choosing $\epsilon = W_+'(x_0)/2$.

Now let us consider more general cases of the function $W_+(x)$ which leads to nonsingular QES potential energy $V_-(x)$.

Case 1. Suppose that $W_+(x)$ has simple zeros at the points x_k, $k = 1, \ldots, n$

$$W_+(x) = W_+'(x_k)(x - x_k) + \tfrac{1}{2}W_+''(x_k)(x - x_k)^2 + \mathrm{O}((x - x_k)^3). \tag{24}$$

The zeros of the function $W_+(x)$ lead to the poles of the function $W_-(x)$ and the superpotential $W(x)$ with the following behaviour in the vicinity of x_k:

$$W(x) = -\left(\frac{1}{2} - \frac{\epsilon}{W_+'(x_k)}\right)\frac{1}{x - x_k} - \frac{1}{2}\frac{W''(x_k)}{W'(x_k)}\left(\frac{1}{2} + \frac{\epsilon}{W_+'(x_k)}\right) + \mathrm{O}(x - x_k). \tag{25}$$

The behaviour of the superpotential $W_1(x)$ in the vicinity of x_k is similar to $W(x)$ only with the opposite sign. It is worth comparing the superpotential (25) with (10): $A_{-1} = \epsilon/W_+'(x_k) - 1/2$ and $A_0 = -W''(x_k)(\epsilon/W_+'(x_k) + 1/2)/2W'(x_k)$. This superpotential leads to the following behaviour of the potential energy in the vicinity of x_k:

$$2V_-(x) = \left[\left(\frac{\epsilon}{W_+'(x_k)}\right)^2 - \frac{1}{4}\right]\left(\frac{1}{(x - x_k)^2} - \frac{W''(x_k)}{W'(x_k)}\frac{1}{(x - x_k)}\right) + \mathrm{O}(\text{const}). \tag{26}$$

Thus, in the case

$$W_+'(x_k) = \pm 2\epsilon \tag{27}$$

the potential energy $V_-(x)$ is free of singularities. It is convenient to divide the set of x_k into two subsets x_k^+ ($k = 1, \ldots, n^+$) and x_k^- ($k = 1, \ldots, n^-$) for which $W_+'(x_k^+) = 2\epsilon > 0$ and $W_+'(x_k^-) = -2\epsilon < 0$. We suppose in this paper that $\epsilon > 0$. Because of $W_+'(x_k^+) = 2\epsilon$ the singularity at the points x_k^+ is cancelled and $W(x)$, $W_1(x)$ have singularities only at the points x_k^-:

$$W(x) = \frac{-1}{x - x_k^-} + \mathrm{O}(x - x_k^-) \tag{28}$$

$$W_1(x) = \frac{1}{x - x_k^-} + \mathrm{O}(x - x_k^-). \tag{29}$$

Substituting $W(x)$ into (8) and using the result of the previous section (see equation (12)) we see that the wavefunction $\psi_0^-(x)$ with zero energy has n^- zeros at the points x_k^-, namely $\psi_0^- \sim (x - x_k^-)$ in the vicinity of x_k^-. Substituting $W_+(x)$ given by (24) and $W_1(x)$ given by (29) into (17) we obtain that the wavefunction $\psi_\epsilon^-(x)$ with the energy ϵ has n^+ zeros at the points x_k^+: $\psi_\epsilon^-(x) \sim (x - x_k^+)$. When $W_+(x)$ is the continuous function satisfying condition (9) then $n^+ = n^- + 1$. Thus, in this case $\psi_0^-(x)$ and $\psi_\epsilon^-(x)$ correspond to n^-th and $(n^- + 1)$th excited states respectively.

Case 2. Now let us assume that the function $W_+(x)$ in addition to the zeros has the simple poles at the points x_k^0 with the behaviour in the vicinity of x_k^0 as follows:

$$W_+(x) = \frac{G_{-1}}{x - x_k^0} + G_0 + \mathrm{O}(x - x_k^0). \tag{30}$$

Then

$$W(x) = \frac{1}{2}\frac{G_{-1}+1}{x - x_k^0} + \frac{1}{2}\frac{G_0}{G_{-1}}(G_{-1} - 1) + \mathrm{O}(x - x_k^0) \tag{31}$$

$$W_1(x) = \frac{1}{2}\frac{G_{-1}-1}{x - x_k^0} + \frac{1}{2}\frac{G_0}{G_{-1}}(G_{-1} + 1) + \mathrm{O}(x - x_k^0). \tag{32}$$

Here we drop the terms of order $(x - x_k^0)$. Note that G_0 and G_{-1} can depend on k. For the sake of simplicity we omit this dependence. Comparing superpotential (31) and (10) we conclude that the superpotential $W(x)$ (31) gives a nonsingular potential energy for the *case 2a*: $G_{-1} = -1$ and G_0 is an arbitrary constant, and for the *case 2b*: $G_{-1} = -3$ and $G_0 = 0$.

Case 2a. For the case $G_{-1} = -1$ in the vicinity of x_k^0 we have nonsingular superpotential $W(x)$ and singular $W_1(x)$

$$W(x) = G_0 + \mathrm{O}(x - x_k^0) \tag{33}$$

$$W_1(x) = \frac{-1}{x - x_k^0} + \mathrm{O}(x - x_k^0). \tag{34}$$

The wavefunctions $\psi_0^-(x)$ and $\psi_\epsilon^-(x)$ calculated with these superpotentials do not have zeros at the points x_k^0, which we denote in this case as a_k, $k = 1, \ldots, n^0$. Nevertheless, now $W_+(x)$ in addition to $n = n^+ + n^-$ zeros at points x_k^+ and x_k^- has n^0 poles at the points x_k^0 and thus is not a continuous function. As a result, in this case we have $n^+ = n^- + n^0 + 1$. Using the result obtained in case 1 we see that $\psi_0^-(x)$ has zeros at x_k^- and corresponds to the n^-th excited state and ψ_ϵ^- has zeros at x_k^+ and corresponds to the $(n^- + n^0 + 1)$th excited state.

Case 2b. The case $G_{-1} = -3$ and $G_0 = 0$ leads to the following behaviour of superpotentials in the vicinity of x_k^0:

$$W(x) = \frac{-1}{x - x_k^0} + \mathrm{O}(x - x_k^0) \tag{35}$$

$$W_1(x) = \frac{-2}{x - x_k^0} + \mathrm{O}(x - x_k^0). \tag{36}$$

The wavefunctions $\psi_0^-(x)$ and $\psi_\epsilon^-(x)$ calculated with these superpotentials have common zeros at the points x_k^0, which we denote in this case as b_k, $k = 1, \ldots, m^0$. Thus, when in addition to poles the function $W_+(x)$ has $n = n^+ + n^-$ zeros at the points x_k^+ and x_k^- the wavefunction $\psi_0^-(x)$ corresponds to the $(n^- + m^0)$th excited state and $\psi_\epsilon^-(x)$ corresponds to the $(n^- + 2m^0 + 1)$th excited state.

Let us consider the *general case* which combines the cases 1, 2a and 2b. The function $W_+(x)$ has n^- zeros with negative derivatives at the points x_k^- ($k = 1, \ldots, n^-$), n^0 poles at the points a_k ($k = 1, \ldots, n^0$) with asymptotic behaviour in the vicinity of these points $-1/(x - a_k)$ + const and m^0 poles at the points b_k ($k = 1, \ldots, m^0$) with the asymptotic behaviour $-3/(x - b_k)$. The number of zeros of the function $W_+(x)$ with a positive derivative at the points x_k^+ is the following: $n^+ = n^- + n^0 + m^0 + 1$. The wavefunction $\psi_0^-(x)$ has $(n^- + m^0)$ nodes at the points x_k^- and b_k and thus corresponds to the $(n^- + m^0)$th excited state. The wavefunction $\psi_\epsilon^-(x)$ has $n^+ + m^0 = n^- + n^0 + 2m^0 + 1$ nodes at the points x_k^+ and b_k and thus corresponds to the $(n^- + n^0 + 2m^0 + 1)$th excited state.

Thus, the considered cases 1, 2a, 2b and the combined general case lead to the nonsingular QES potential energy $V_-(x)$ given by (3), where the superpotential $W(x)$ is expressed over the function $W_+(x)$ by equation (22). The zero-energy wavefunction $\psi_0^-(x)$ and the wavefunction $\psi_\epsilon^-(x)$ with energy ϵ are given by (8) and (17), respectively. Note that in the case of nonsingular superpotential $W(x)$ the zero-energy wavefunction corresponds to the ground state, but in the case of singular superpotential the energy of the ground state is less than zero and the zero-energy wavefunction corresponds to an excited state.

To conclude this section let us discuss the second possibility for construction of the QES potentials with two known eigenstates: namely, use of the solution of equation (18) with respect to $W_+(x)$ [21]. We found that

$$W_-(x) = -\frac{\phi''(x)}{\phi'(x)} \qquad W_+(x) = 2\epsilon\frac{\phi(x)}{\phi'(x)} \tag{37}$$

satisfies equation (18), where $\phi(x)$ is new generating function. Using W_+ and W_- given by (37) we obtained the QES potential and two wavefunctions in terms of $\phi(x)$. Choosing generating functions $\phi(x)$ with one zero we obtained QES potentials with explicitly known ground and first excited states. Note that the basic equations derived by Dolya and Zaslavskii in [19] without resorting to SUSY quantum mechanics are the same as earlier obtained in [21] using the SUSY method (in [19] $\phi(x)$ is denoted as $\xi(x)$). A new result obtained by Dolya and Zaslavskii is that they have shown how one can obtain not only the ground and first excited states but any pair of states using generating function $\phi(x)$ and derivative $\phi'(x)$ with zeros and poles. In this paper we work with generating function $W_+(x)$ which is related with $\phi(x)$ by (37). As we see, zeros and poles of $\phi(x)$ (or $\xi(x)$) lead the zeros of $W_+(x)$ and zeros of $\phi'(x)$ (or $\xi'(x)$) lead to the poles of $W_+(x)$.

4. Examples of QES potentials

Note that all expressions depend on the function $W_+(x)$. We may choose various functions $W_+(x)$ and obtain as a result various QES potentials. Note also that when the function $W_+(x)$ generates the potential energy $V_-(x)$ then $W_+(x/a)/a$ generates the potential energy $V_-(x/a)/a^2$. This scaling is useful for comparing, in principle, the same potential energies written in different forms as a result of different measurement units.

To illustrate the above-described method we give two explicit examples of the nonsingular QES potentials.

Example 1. Let us consider a continuous function $W_+(x)$ which corresponds to case 1:

$$W_+(x) = \alpha x\frac{x^2 - 1}{x^2 + 1} \tag{38}$$

where $\alpha > 0$. This function has three zeros at the points $0, \pm 1$. The denominator is written in order to satisfy condition (27), namely $W'_+(0) = -\alpha$, $W'_+(\pm 1) = \alpha$. Note that $n^+ = 2$, $n^- = 1$ and thus we have QES potential with explicitly known first and second excited states. From the condition of nonsingularity of the potential energy it follows that $\epsilon = \alpha/2$. Then using (22) and (23) we obtain for the superpotentials

$$W(x) = \frac{\alpha}{2}x + (1-\alpha)\frac{x}{x^2+1} - \frac{1}{x} \tag{39}$$

$$W_1(x) = \frac{\alpha}{2}x - (1+\alpha)\frac{x}{x^2+1} + \frac{1}{x}. \tag{40}$$

The superpotential $W(x)$ gives the following QES potential:

$$2V_-(x) = \frac{\alpha^2}{4}x^2 - (1-\alpha)(3-\alpha)\frac{1}{(x^2+1)^2} - 2\alpha(1-\alpha)\frac{1}{x^2+1} + \alpha(1-\alpha) - \frac{3}{2}\alpha. \tag{41}$$

The zero-energy wavefunction (8) and wavefunction with energy $\epsilon = \alpha/2$ (17) read

$$\psi_0^-(x) = C_0 x(x^2+1)^{(\alpha-1)/2}\exp\left(-\frac{\alpha}{4}x^2\right) \tag{42}$$

$$\psi_\epsilon^-(x) = C_\epsilon(x^2-1)(x^2+1)^{(\alpha-1)/2}\exp\left(-\frac{\alpha}{4}x^2\right). \tag{43}$$

As we see, $\psi_0^-(x)$ has one node and thus really corresponds to the first excited state; $\psi_\epsilon^-(x)$ has two nodes and corresponds to the second excited state.

Note that QES potential (41) has similar structure as the potential studied in [19] but in fact it is another potential. In our case we have the QES potential with explicitly known first and second eigenstates, whereas for the QES potential studied in [19] the ground and second excited states are explicitly known. It is interesting to note also that in our case QES potential (41) at the value of parameter $\alpha = 1$ becomes exactly solvable and corresponds to the harmonic oscillator.

Example 2. Let us consider a more complicated example for which the function $W_+(x)$ has two poles and three zeros:

$$W_+(x) = \frac{\alpha}{x^2-1}x(x^2-a^2)(x^2+b^2). \tag{44}$$

In order to have the asymptotic behaviour of the function $W_+(x)$ in the vicinity of the points $x = \pm 1$ which corresponds to the case 2a, namely $-1/(x-1)$ and $-1/(x+1)$, we choose the parameter α as follows:

$$\alpha = \frac{2}{(a^2-1)(b^2+1)}. \tag{45}$$

This value of α gives a nonsingular behaviour of the potential energy in the vicinity of $x = \pm 1$. In order to have the same derivatives in the points of zeros of the function $W_+(x)$, namely $W'_+(0) = W'_+(\pm a)$, we put the following value for parameter b:

$$b^2 = \frac{2a^2}{(a^2-3)} \tag{46}$$

from which it follows that $a^2 > 3$. Then, choosing

$$2\epsilon = W'_+(0) = W'_+(\pm a) = \frac{4a^4}{3(a^2-1)^2} \tag{47}$$

we obtain the superpotentials

$$W(x) = \frac{x}{6}\left(-\frac{2(a^4-6a^2+3)}{(a^2-1)^2} + \frac{2(a^2-3)x^2}{(a^2-1)^2} - \frac{15(a^2-3)}{(a^2-3)x^2+2a^2}\right) \tag{48}$$

$$W_1(x) = \frac{x}{6}\left(-\frac{2(a^4-6a^2+3)}{(a^2-1)^2} + \frac{2(a^2-3)x^2}{(a^2-1)^2} + \frac{15(a^2-3)}{(a^2-3)x^2+2a^2} - \frac{12}{x^2-1}\right) \tag{49}$$

and the nonsingular potential energy

$$\begin{aligned} 2V_-(x) = {} & \frac{x^2}{36}\left(-\frac{2(a^4-6a^2+3)}{(a^2-1)^2} + \frac{2(a^2-3)x^2}{(a^2-1)^2} - \frac{15(a^2-3)}{(a^2-3)x^2+2a^2}\right)^2 \\ & + \frac{1}{6}\left(\frac{2(a^4-6a^2+3)}{(a^2-1)^2} - \frac{6(a^2-3)x^2}{(a^2-1)^2} + \frac{15(a^2-3)}{(a^2-3)x^2+2a^2}\right. \\ & \left. - \frac{30(a^2-3)^2x^2}{((a^2-3)x^2+2a^2)^2}\right). \end{aligned} \tag{50}$$

For this QES potential we know explicitly the wavefunctions of the ground and the third excited states

$$\psi_0^-(x) = C_0((a^2-3)x^2+2a^2)^{5/4}\exp\left(\frac{-(a^2-3)x^4+2(a^4-6a^2+3)x^2}{12(a^2-1)^2}\right) \tag{51}$$

$$\psi_\epsilon^-(x) = C_\epsilon\frac{x(x^2-a^2)}{((a^2-3)x^2+2a^2)^{1/4}}\exp\left(\frac{-(a^2-3)x^4+2(a^4-6a^2+3)x^2}{12(a^2-1)^2}\right). \tag{52}$$

Note that at $a^2 = 3$ this potential becomes exactly solvable and corresponds to the harmonic oscillator.

5. Conclusions

We propose the SUSY method for constructing the QES potentials with arbitrary two known energy levels and corresponding wavefunctions. This is an extension of our SUSY method proposed in [17, 21] where QES potentials with the ground and first excited states were obtained. In the proposed method the function $W_+(x)$ plays the role of a generating function. Choosing different functions $W_+(x)$ we obtain different QES potentials $V_-(x)$. The two known wavefunctions $\psi_0^-(x)$ and $\psi_\epsilon^-(x)$ correspond to the eigenstates with zero energy and energy ϵ, respectively. The state numbers of these wavefunctions depend on the properties of function $W_+(x)$ as described in the cases 1, 2a, 2b which is summarized in the general case in section 3. In section 4 we consider the explicit examples of QES potentials with the rational generating function $W_+(x)$. These examples yield some new one-parametric QES potentials. At some special values of parameters these potentials become exactly solvable and correspond to the harmonic oscillator.

We can consider various new generating functions $W_+(x)$ and obtain new QES potentials. One of the interesting possibilities is to consider the periodic functions $W_+(x)$ which lead to the periodic QES potentials. This problem will be the subject of a separate paper.

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